Cauchy - Goursat Theorem

in the case when the holomorphic fcn. is C^1: α is a positively oriented simple, closed, rectifiable curve, $f(z)$ is holomorphic and C^1. Also, $\text{interior}(\alpha) \subseteq D$ ← domain of f

Green's Thm proof:

differentials: f is a fcn on $\mathbb{R}^2$,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

(where $dz = dx + i\,dy$, $d\bar{z} = dx - i\,dy$,

$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$)

Holomorphic case: $\frac{\partial f}{\partial \bar{z}} = 0$

$$\Rightarrow df = \frac{\partial f}{\partial z} dz = f'(z) dz$$

Green's Thm: β is a 1-form (that is C^1)

$$\oint_\alpha \beta = \int_{\text{interior}(\alpha)} d\beta$$

$$\Rightarrow \oint_\alpha f(z) dz = \int_{\text{interior}(\alpha)} d(f(z)\, dz)$$

$$= \int_{\text{interior}(\alpha)} d(f(z)) \wedge dz$$

$$= \int_{\text{interior}(\alpha)} f'(z) \underbrace{dz \wedge dz}_{0} = 0. \quad \blacksquare$$

More about integrals over curves α

① γ is an oriented curve

$-\gamma$ is defined to be the same curve but directed in the opposite direction.

Observe $\displaystyle\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz.$

② $\displaystyle\left|\int_{\gamma} f(z)\,dz\right| \leq \int_{\gamma} |f(z)|\,|dz|$

If $|f(z)| \leq M$ for $z \in \gamma$, and $L = \text{length}(\gamma)$,

then $\displaystyle\left|\int_{\gamma} f(z)\,dz\right| \leq \int_{\gamma} |f(z)|\,|dz|$

$$\leq \int_{\gamma} M\,|dz| = M\int_{\gamma} |dz| = ML.$$

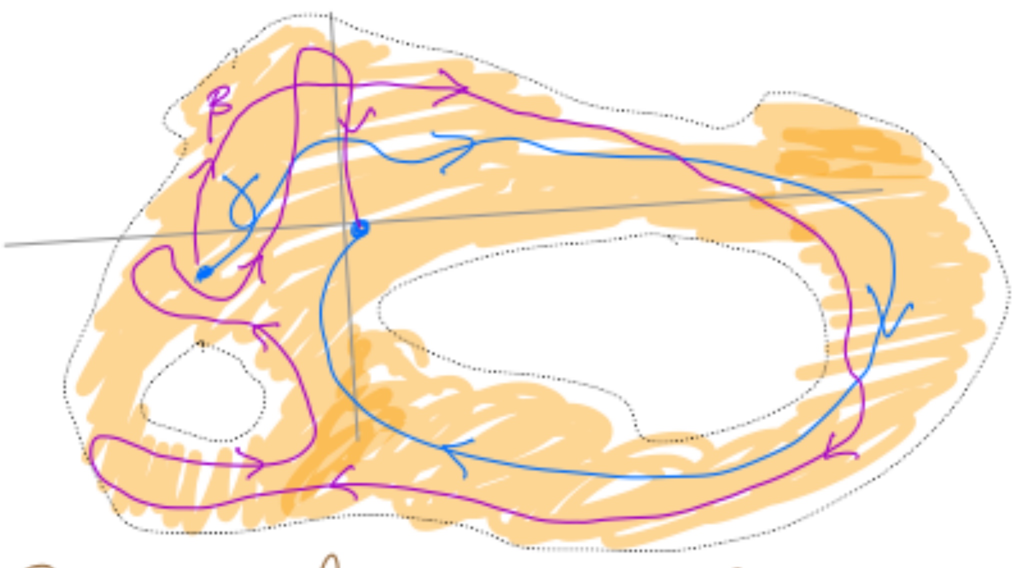
<u>Deformation Theorem</u> If γ is a curve in domain(f), with f holomorphic, and if it can be deformed to a curve β without leaving the domain of f, then

$$\int_\gamma f = \int_\beta f$$ (if γ & β have the same starting & ending points)

The last hypothesis is not necessary if γ, β are closed curves.

idea of the proof:

① Non-closed curves:

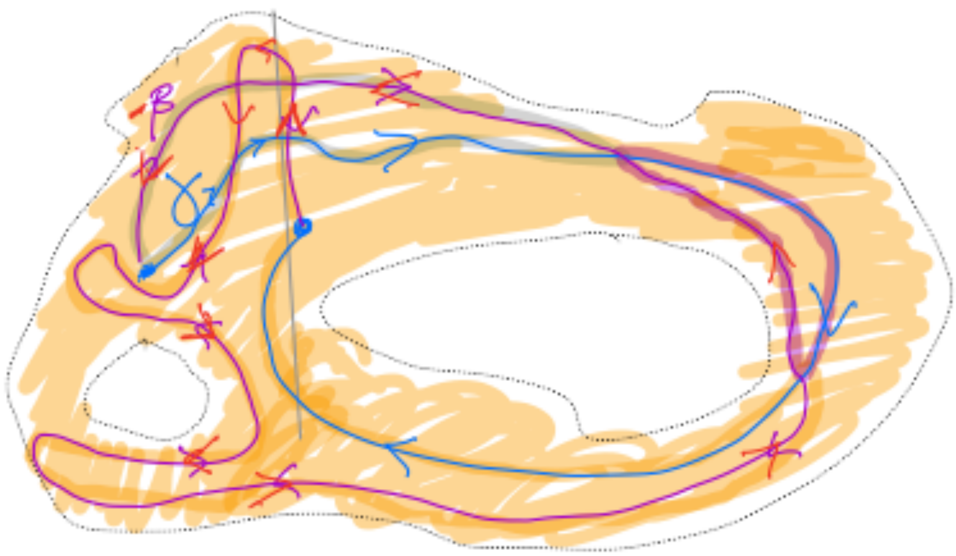


Consider $\int_\gamma f - \int_\beta f = \int_{\gamma \cup (-\beta)} f$

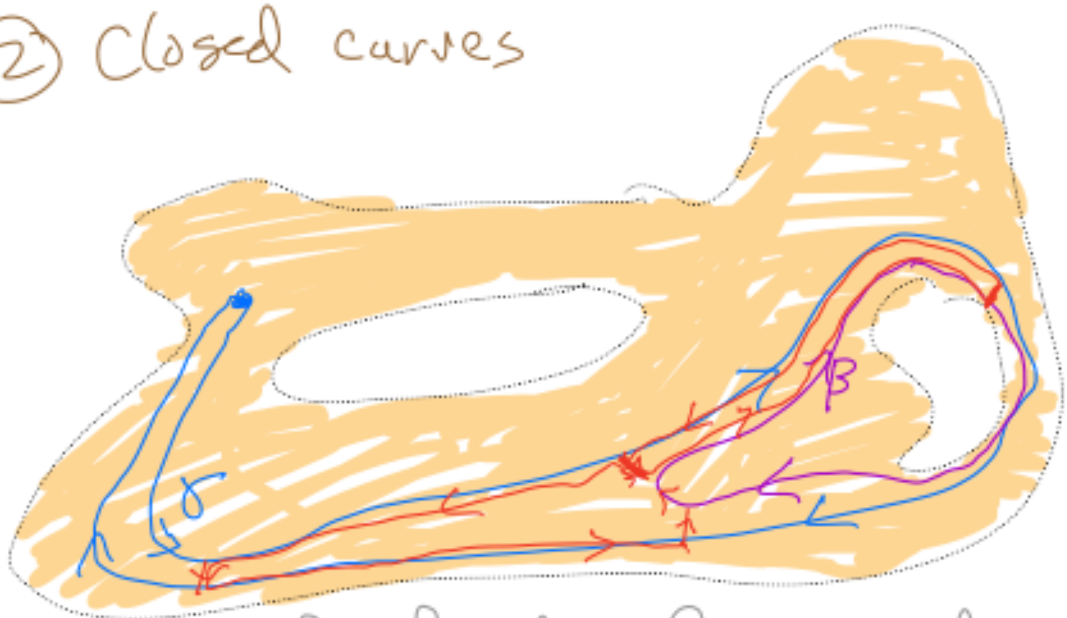
closed curve

Can be decomposed into simple closed curve pieces, so $\int_{\gamma \cup (-\beta)} f = 0$

by Cauchy's Thm.



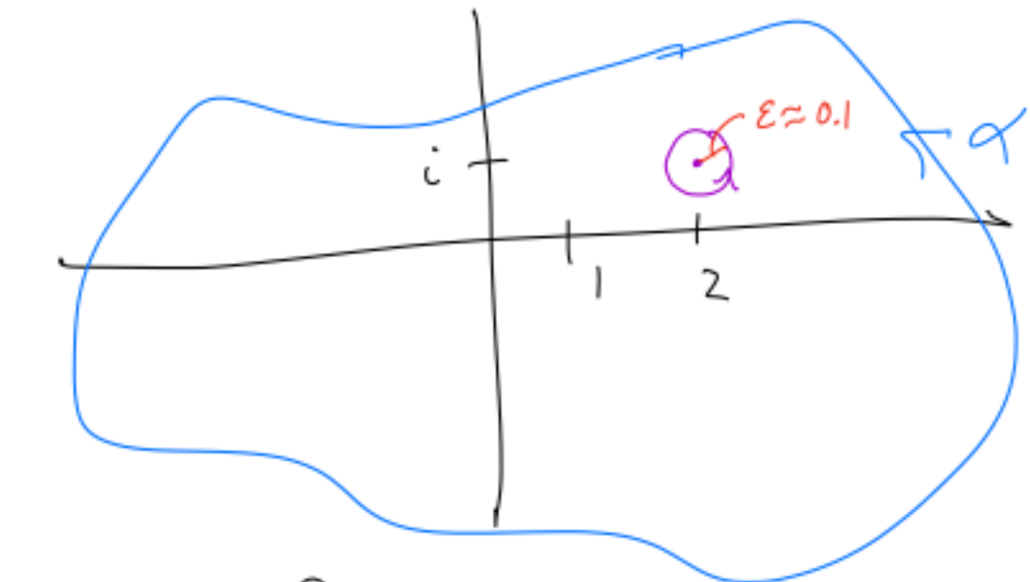
② Closed curves



Consider: $\int_{-\gamma \cup \beta} f(z)\,dz = \int_{-\gamma} f(z)\,dz + \int_{\beta} f(z)\,dz$

$$= -\int_\gamma f(z)\,dz + \int_\beta f(z)\,dz$$

$$= 0$$

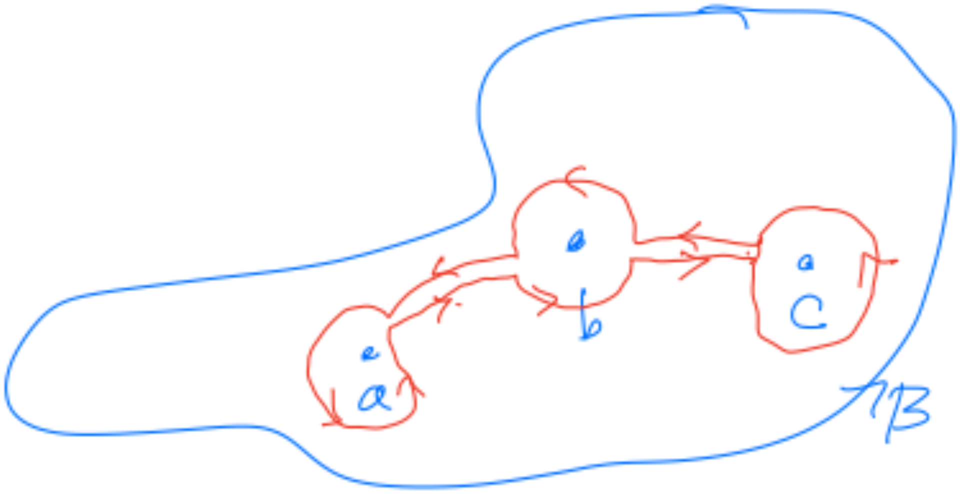


$$\int_\alpha \frac{1}{z-2-i}\,dz = 2\pi i$$

Why? $\displaystyle\int_\alpha \frac{1}{z-2-i}\,dz = \int_{\partial B(2+i,\varepsilon)} \frac{1}{z-(2+i)}\,dz$

$= 2\pi i$.

by the Deformation Theorem.



$$\int_{\beta}\left(\frac{1}{z-a}+\frac{1}{z-b}-\frac{4i}{z-c}\right)dz$$

$$=2\pi i+2\pi i+(-4i)(2\pi i).$$